PRIMITIV XARAKTERLAR QIYMATLARI VA GAUSS YIG'INDILARI ORASIDAGI BOG'LIQLIK.

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Annotatsiya.

Ushbu ishda Primitiv xarakterlar qiymatlari bilan va Gauss yig'indilari qiymatlari orasidagi bog'lanish o'rganilgan, yani agar $\chi(n)$ q moduli bo'yicha primitiv xarakter, $S(q;a,\chi)=\sum_{n=1}^{q}\chi(n)e\left(\frac{an}{q}\right)$ Gauss yig'indisi bo'lsa u holda $\tau(\bar{\chi})\chi(n)=\sum_{a=1}^{q}\bar{\chi}(a)e(\frac{an}{q})$ tenglikning o'rinli ekanligi isbotlangan.

Tayanch so'zlar: Primitiv xarakter, Dirixle xarakteri, Xarakterning darajasi, Kompleks xarakter, Chegirmalar sinflari, Kvadratik chegirma, Indekslar, taqqoslama, Lejandr simvoli

1. **Kirish.** Avvalo xarakterlar va ularning xossalari haqida batafsil to'xtab o'tamiz. Faraz qilaylik, $p>2$ – tub son, $q=p^{\alpha}$, $\alpha\geq 1$ – butun son bo'lsin. Bizga ma'lumki, bunday q moduli bo'yicha boshlang'ich ildizlar mavjud. g ularning eng kichigi bo'lsin. $indn$ bilan $(n,q)=1$ shartni qanoatlantiruvchi n soning q moduli bo'yicha g asosga ko'ra indeksini belgilaymiz, ya'ni $\gamma=\gamma(n)=indn$ soni [1,2,3]

$$g^{\gamma}\equiv n\ (mod q)$$

taqqoslamadan aniqlanadi.

1-ta'rif. $q=p^{\alpha}$ ($q>2$ – tub son, $\alpha\geq 1$ – butun son) moduli bo'yicha xarakter deb aniqlanish sohasi butun sonlardan iborat

$$\chi(n)=\chi(n;q)=\chi(n;q;m)=\begin{cases}0, & \text{agar } (n,q)>1 \text{ bo'lsa}\\ e^{2\pi i\frac{m\cdot indn}{\varphi(q)}}, & \text{agar } (n,q)=1 \text{ bo'lsa}\end{cases}$$

tenglik bilan aniqlanuvchi funksiyaga aytiladi. Bunda m –butun son, $\varphi(q)-$ Eyler funksiyasi.

2-ta'rif. $q=2^{\alpha}$ ($\alpha\geq 1$ – butun son) moduli bo'yicha xarakter deb aniqlanish sohasi butun sonlardan iborat quyidagi tengliklarning biri bilan aniqlanuvchi $\chi(n)$ funksiyaga aytiladi:

$$\chi(n)=\chi(n;2)=\chi(n;2;0;0)=\begin{cases}0, & \text{agar } (n,2)>1 \text{ bo'lsa}\\ 1, & \text{agar } (n,2)=1 \text{ bo'lsa,}\end{cases}$$

$$\chi(n)=\chi(n;4)=\chi(n;4;m_0;0)=\begin{cases}0, & \text{agar } (n,4)>1 \text{ bo'lsa}\\ (-1)^{m_0\gamma_0}, & \text{agar } (n,4)=1 \text{ bo'lsa}\end{cases}$$

bunda $n\equiv(-1)^{\gamma_0}mod4)$, m_0- butun son;

$$\chi(n)=\chi(n;2^{\alpha})=\chi(n;2^{\alpha};m_0;m_1)$$
$$=\begin{cases}0, & \text{agar } (n,2^{\alpha})>1 \text{ bo'lsa}\\ (-1)^{m_0\gamma_0}e^{2\pi i\frac{m_1\gamma_1}{2^{\alpha-2}}}, & \text{agar } (n,2^{\alpha})=1, \quad \alpha\geq 3-\text{bo'lsa},\end{cases}$$

bu yerda m_0, m_1lar butun sonlar.

Ta'rifga ko'ra $\chi(n)=\chi(n;q;m)-$ xarakter m parametrga bog'liq, m bo'yicha davriy bo'lib davri $\varphi(q)$ ga teng, ya'ni umuman aytganda q moduli bo'yicha $\varphi(q)$ ta xarakter mavjud bo'lib ularni $m=0,1,2,\dots,\varphi(q)-1$ deb olib hosil qilish mumkin.

Umuman olganda $\chi(n)$ xarakterning eng kichik davri uning modulidan kichik bo'lishi mumkin. Ko'pchilik tekshirishlarda primitiv (boshlang'ich) xarakterlar deb ataluvchi eng kichik davri uning moduliga teng xarakterlar muhim ahamiyatga ega.

3-ta'rif. Agar $(m,q)=1$ bo'lsa, u holda $q=p^{\alpha}$ ($p>2$ – tub son) moduli bo'yicha bosh xarakterga teng bo'lmagan $\chi(n)=\chi(n;q,m)$ xarakterga *primitiv xarakter* deyiladi;

agarda $m_0=1,(m_1,2)=1$ bo'lsa, $q=2^{\alpha}$, ($\alpha\geq 3$ – butun son) moduli bo'yicha bosh xarakterdan farqli $\chi(n)=\chi(n;q)=\chi(n;q;m_0;m_1)$ xarakterga primitiv xarakter deyiladi;

4 moduli bo'yicha bosh xarakterdan farqli $\chi(n)=\chi(n;q;m)$ xarakterga primitiv xarakter deyiladi.

Shu bilan birga $\chi(n)$ multiplikativ funksiya, ya'ni $\chi(n\cdot m)=\chi(n)\cdot\chi(m)$ hamda trigonometrik yig'indilar uchun ushbu tenglik o'rinli:

$$\frac{1}{m}\sum_{x=0}^{m-1}e^{2\pi i\frac{ax}{m}}=\begin{cases}0, \text{ agar } & a\not\equiv 0(mod\ m) \quad \text{bo'lsa;}\\ 1, \text{ agar } & a\equiv 0(mod\ m) \quad \text{bo'lsa.}\end{cases} \tag{1}$$

2. **Asosiy qism.** Primitiv xarakterlar uchun maxsus formula mavjud bo'lib, u primitiv xarakterlar qiymatlari bilan va Gauss yig'indilari qiymatlari orasidagi bog'lanishni ifodalaydi [2,3]. Bizga ma'lumki Gauss yig'indisi quyidagi ko'rinishda ifodalanadi:

$$S=S(q;a,\chi)=\sum_{n=1}^{q}\chi(n)e\left(\frac{an}{q}\right).$$

Ushbu ishning asosiy natijasini quyidagi teorema orqali bayon qilamiz.

1-teorema. Agar $\chi(n)$ q moduli bo'yicha primitiv xarakter bo'lsa u holda

$$\tau(\bar{\chi})\chi(n) = \sum_{a=1}^{q} \bar{\chi}(a) e^{2\pi i \frac{an}{q}} , \qquad (2)$$

bu yerda

$$\tau(\chi) = \sum_{a=1}^{q} \chi(a) e^{2\pi i \frac{a}{q}} , \qquad |\tau(\chi)| = \sqrt{q} . \qquad (3)$$

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Isboti. $q = 4$ bo'lganda (2) va (3) tengliklarni bevosita tekshirib ko'rish mumkin. $q \neq 4$ va $(n, q) = 1$ bo'lsin. U holda m sonini $mn \equiv 1 (mod\, q)$ taqqoslamadan aniqlab
1). $\chi(n) = \chi(m)$ yani xarakterlarning davriyligini 2). χ multiplikativ xarakter ekanligi, 3). agar a soni q moduli bo'yicha chegirmalarning to'la sistemasini qabul qilsa, u holda am ham shu sistemani qabul qilishini hisobga olib quyidagiga ega bo'lamiz [4, 5]:

$$\chi(n)\tau(\bar{\chi}) = \bar{\chi}(m) \sum_{a=1}^{q} \bar{\chi}(a) e\left(\frac{a}{q}\right) = \sum_{a=1}^{q} \bar{\chi}(am) e\left(\frac{amn}{q}\right) = \sum_{a=1}^{q} \bar{\chi}(a) e\left(\frac{an}{q}\right).$$

Endi $(n, q) > 1$ holni qaraymiz. Bu holda (2) ning chap tomoni

$$\tau(\bar{\chi})\chi(n) = \sum_{a=1}^{k} \bar{\chi}(a) e\left(\frac{an}{q}\right)$$

nolga teng. Agar $q = p > 2$ bo'lsa, u holda $(n, q) = (n, p) = p$ va (2) ning o'ng tomoni $\chi(n) \neq \chi_0(n)$ va

$$\sum_{a=1}^{p} \bar{\chi}(a) e\left(\frac{an}{p}\right) = \sum_{a=1}^{p} \bar{\chi}(a) = 0$$

bo'lgani uchun nolga teng.

Endi faraz etaylik $q = p^{\alpha}$, $\alpha > 1$, $n = rp$ bo'lsin. U holda Gauss yig'indisi quyidagiga teng:

$$S = \sum_{a=1}^{p^{\alpha}} \bar{\chi}(a) e\left(\frac{arp}{p^{\alpha}}\right) = \sum_{a=1}^{p^{\alpha}} \bar{\chi}(a) e\left(\frac{ar}{p^{\alpha-1}}\right).$$

Bu yerda yig'indida a, $1 \leq a \leq p^{\alpha}$ o'zgaruvchini quyidagicha almashtiramiz

$a = up^{\alpha-1} + v$ bu yerda $0 \le u \le p-1$, $1 \le v \le p^{\alpha-1}$. Natijada bu yig'indi

$$S = \sum_{u=0}^{p-1} \sum_{\substack{v=1 \\ (v,p)=1}}^{p^{\alpha-1}} \bar{\chi}(up^{\alpha-1}+v) e\left(\frac{r(up^{\alpha-1}+v)}{p^{\alpha}}\right) =$$

$$= \sum_{\substack{v=1 \\ (v,p)=1}}^{p^{\alpha-1}} e\left(\frac{rv}{p^{\alpha-1}}\right) \sum_{u=0}^{p-1} \bar{\chi}(up^{\alpha-1}+v)$$

ko'rinishga ega bo'ladi. Bundan $(v,p)=1$ ekanligini hisobga olsak, shunday $v_1 (v_1, p) = 1$ mavjudki, $vv_1 \equiv 1\ (mod\ q\ )$ va $\chi(vv_1) = 1$. Shu sababli:

$$S = \sum_{\substack{v=1 \\ (v,p)=1}}^{p^{\alpha-1}} \chi(vv_1) e\left(\frac{rv}{p^{\alpha-1}}\right) \sum_{u=0}^{p-1} \bar{\chi}(up^{\alpha-1}+v) =$$

$$= \sum_{\substack{v=1 \\ (v,p)=1}}^{p^{\alpha-1}} \chi(v) e\left(\frac{rv}{p^{\alpha-1}}\right) \sum_{u=0}^{p-1} \bar{\chi}(uv_1 p^{\alpha-1}+vv_1) =$$

$$= \sum_{\substack{v=1 \\ (v,p)=1}}^{p^{\alpha-1}} \chi(v) e\left(\frac{rv}{p^{\alpha-1}}\right) \sum_{u=0}^{p-1} \bar{\chi}(uv_1 p^{\alpha-1}+1).$$

Agar u chegirmalarning to'la sistemasini tashkil etsa, $v_1 u$ ham to'liq sistemani tashkil etadi. Shunday qilib, umumiylikka zarar bermagan holda ichki yig'indida $v_1 u$ ni yana u bilan belgilab, quyidagini hosil qilamiz:

$$S = \sum_{\substack{v=1 \\ (v,p)=1}}^{p^{\alpha-1}} \chi(v) e\left(\frac{rv}{p^{\alpha-1}}\right) \sum_{u=0}^{p-1} \bar{\chi}(up^{\alpha-1}+1).$$

Endi biz

$$\sum_{u=0}^{p-1} \bar{\chi}\,(up^{\alpha-1}+1) = 0$$

tenglikning to'g'ri ekanligini ko'psatish yetarli. Aytaylik $p > 2$ va g esa p modul bo'yicha boshlangich ildiz bo'lsin. U holda shunday t mavjud bo'lib $g + pt$ soni har qanday $\alpha > 1$ uchun p^{α} moduli bo'yicha boshlang'ich ildiz bo'ladi. Bu yerda t

$$(g + pt\,)^{p-1} = 1 + pb, \qquad (b,p) = 1$$

shartni qanoatlantiradi. Agar γ soni $up^{\alpha-1}+1$ sonining p^{α} moduli bo'yicha indeksi bo'lsa, u holda γ=$(p-1)\gamma_1$;

$$(g+pt)^{\gamma}=(1+pb)^{\gamma_1}\equiv up^{\alpha-1}+1(mod p^{\alpha}).$$

Bu yerdan $\gamma_1=ub_1p^{\alpha-2}$, $bb_1\equiv 1(mod p)$ kelib chiqadi.

Shunday qilib,

$$\bar{\chi}(up^{\alpha-1}+1)=e^{-2\pi i\frac{m\, ind(up^{\alpha-1}+1)}{\varphi(p^{\alpha})}}=e^{-2\pi i\frac{mub_1}{p}},$$

bu yerda $(mb_1,p)=1$ va

$$\sum_{u=0}^{p-1}e^{-2\pi i\frac{mub_1}{p}}=0.$$

Endi faraz etaylik $p=2$, $q=2^{\alpha}$, $\alpha\geq 3$ bo'lsin. U holda $u2^{\alpha-1}+1$ sonining indekslari 0, $2^{\alpha-3}$ ga teng, shuning uchun ham $(m_0=1,\ (m_1,2)=1)$

$$\sum_{u=0}^{1}\bar{\chi}(1+u2^{\alpha-1})=1+(-1)^0 e(\frac{m_1 2^{\alpha-3}}{2^{\alpha-2}})=0$$

bo'ladi. Shunday qilib (2)-tenglik ixtiyoriy *n* uchun isbotlandi.

(1) va (2) lardan

$$\varphi(q)|\tau(\bar{\chi})|^2=|\tau(\bar{\chi})|^2\sum_{n=1}^{q}|\chi(n)|^2=\sum_{n=1}^{q}\left|\sum_{a=1}^{k}\bar{\chi}(a)e^{2\pi i\frac{an}{q}}\right|^2=$$

$$=\sum_{a,b=1}^{q}\bar{\chi}(a)\chi(b)\sum_{n=1}^{k,}e^{2\pi i\frac{(a-b)n}{q}}=q\varphi(q),$$

ya'ni $|\tau(\bar{\chi})|^2=q$ kelib chiqadi. Bundan esa (3) ga ega bo'lamiz. Lemma to'la isbot bo'ldi.

Foydalanilgan adabiyotlar.

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